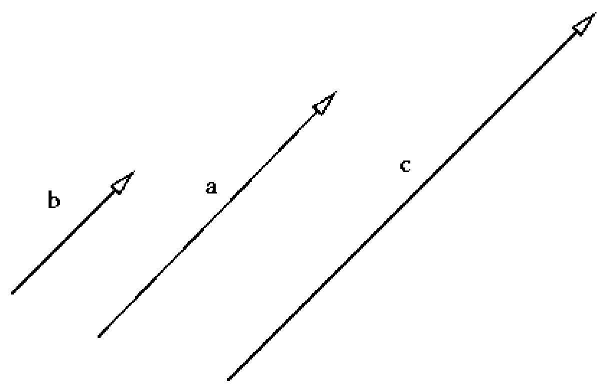
More Practice Problems

MA2160

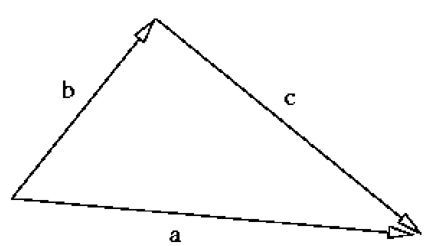
February 14, 2007

1. Graph three vectors $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$, such that

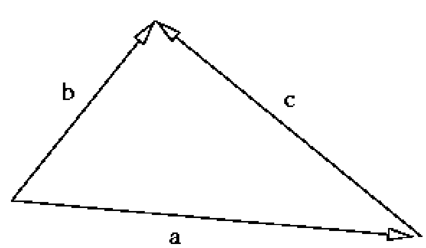
   (a) $\overrightarrow{a} = 2\overrightarrow{b}$ and $\overrightarrow{c} = 3\overrightarrow{b}$:



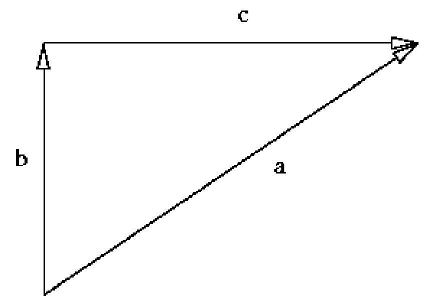
   (b) $\overrightarrow{a} = \overrightarrow{b} + \overrightarrow{c}$, where $\overrightarrow{b}$ and $\overrightarrow{c}$ are not parallel:



   (c) $\overrightarrow{a} = \overrightarrow{b} - \overrightarrow{c}$, where $\overrightarrow{b}$ and $\overrightarrow{c}$ are not parallel:



   (d) $\overrightarrow{a} = \overrightarrow{b} + \overrightarrow{c}$ and $\overrightarrow{b} \cdot \overrightarrow{c} = 0$:



2. Consider the vectors

$$\begin{aligned}\overrightarrow{a} &= 3\hat{\imath} + \hat{\jmath} - 2\hat{k}\\ \overrightarrow{b} &= -2\hat{\imath} - 3\hat{\jmath} - 2\hat{k}\\ \overrightarrow{c} &= -6\hat{\imath} - 2\hat{\jmath} + 4\hat{k}\\ \overrightarrow{d} &= -\hat{\imath} + 2\hat{\jmath}\\ \overrightarrow{e} &= \hat{\imath} - \hat{\jmath} + \hat{k}\\ \overrightarrow{f} &= -2\hat{\imath} - 1\hat{\jmath} + 3\hat{k}\\ \overrightarrow{g} &= 3\hat{\imath} - \hat{\jmath} - 2\hat{k}\end{aligned}$$

(a) $\overrightarrow{a} + \overrightarrow{b} = 3\overrightarrow{i} + \overrightarrow{j} - 2\overrightarrow{k} - 2\overrightarrow{i} - 3\overrightarrow{j} - 2\overrightarrow{k} = \overrightarrow{i} - 2\overrightarrow{j} - 4\overrightarrow{k}$

(b) $\overrightarrow{a} \cdot \overrightarrow{b} = 3(-2) + 1(-3) + (-2)(-2) = -6 - 3 + 4 = -5$

(c) $\overrightarrow{a} \times \overrightarrow{b} = (1(-2) - (-2)(-3))\,\overrightarrow{i} + ((-2)(-2) - 3(-2))\,\overrightarrow{j} + (3(-3) - 1(-2))\,\overrightarrow{k}$
$= (-2 - 6)\,\overrightarrow{i} + (4 + 6))\,\overrightarrow{j} + (-9 + 2))\,\overrightarrow{k} = -8\overrightarrow{i} + 10\overrightarrow{j} - 7\overrightarrow{k}$

(d) $3\overrightarrow{c} \cdot (-\overrightarrow{d}) = -3\left((-6)(-1) + (-2)(2) + 4(0)\right) = -3 \cdot 2 = -6$

(e) $3\overrightarrow{e} - 2\overrightarrow{f} + \overrightarrow{c} = 3\overrightarrow{i} - 3\overrightarrow{j} + 3\overrightarrow{k} + 4\overrightarrow{i} + 2\overrightarrow{j} - 6\overrightarrow{k} - 6\overrightarrow{i} - 2\overrightarrow{j} + 4\overrightarrow{k}$
$\overrightarrow{i} - 3\overrightarrow{j} + \overrightarrow{k}$

(f) $||\overrightarrow{c}|| = \sqrt{(-6)^2 + (-2)^2 + 4^2} = \sqrt{56}$

(g) $||12\overrightarrow{f}|| = 12||\overrightarrow{f}|| = 12\sqrt{(-2)^2 + (-1)^2 + 3^2)} = 12\sqrt{14}$

(h) $\overrightarrow{a} + \overrightarrow{c} = 3\overrightarrow{i} + \overrightarrow{j} - 2\overrightarrow{k} - 6\overrightarrow{i} - 2\overrightarrow{j} + 4\overrightarrow{k} = -3\overrightarrow{i} - 1\overrightarrow{j} + 2\overrightarrow{k}$
$(\overrightarrow{a} + \overrightarrow{c}) \cdot \overrightarrow{d} = (-3)(-1) + (-1)(2) + (2)(0) = 3 - 2 = 1$

(i) $\overrightarrow{b} - \overrightarrow{d} = -2\overrightarrow{i} - 3\overrightarrow{j} - 2\overrightarrow{k} - \overrightarrow{i} + 2\overrightarrow{j} = -\overrightarrow{i} - 5\overrightarrow{j} - 2\overrightarrow{k}$
$(\overrightarrow{b} - \overrightarrow{d}) \times \overrightarrow{a} = ((-5)(-2) - (-2)(1))\,\overrightarrow{i} + ((-2)(3) - (-1)(-2))\,\overrightarrow{j}$
$+((-1)(1) - (-5)(3))\,\overrightarrow{k} = (10 + 2)\,\overrightarrow{i} + (-6 - 2))\,\overrightarrow{j} + (-1 + 15))\,\overrightarrow{k} = 12\overrightarrow{i} - 8\overrightarrow{j} + 14\overrightarrow{k}$

(j) $-(\overrightarrow{b} \times \overrightarrow{a}) = -\overrightarrow{a} \times \overrightarrow{b} = -8\overrightarrow{i} + 10\overrightarrow{j} - 7\overrightarrow{k}$

(k) $\overrightarrow{a}$ and $\overrightarrow{f}$ are not parallel to each other, since they are not scalar multiples of each other.

(l) $\overrightarrow{a}$ and $\overrightarrow{c}$ are parallel to each other, since $\overrightarrow{c} = -2\overrightarrow{a}$

(m) $\overrightarrow{b}$ and $\overrightarrow{f}$ are not perpendicular, since $\overrightarrow{b} \cdot \overrightarrow{f} = 1 \neq 0$.

(n) $\overrightarrow{a}$ and $\overrightarrow{c}$ are perpendicular to $\overrightarrow{e}$, since $\overrightarrow{a} \cdot \overrightarrow{e} = \overrightarrow{c} \cdot \overrightarrow{e} = 0$.

(o) The unit vector in direction of $\overrightarrow{c}$ is $\overrightarrow{u} = \frac{\overrightarrow{c}}{||\overrightarrow{c}||}$. Therefore a vector with length 10 parallel to $\overrightarrow{c}$ can be found as $10\overrightarrow{u} = 10\frac{\overrightarrow{c}}{||\overrightarrow{c}||} = \frac{10}{\sqrt{56}}\overrightarrow{c}$.

(p) If $\lambda = -1$, then $\overrightarrow{c} = -2\overrightarrow{g}$ and therefore they are parallel. $\overrightarrow{c}$ and $\overrightarrow{g}$ are perpendicular, if $\overrightarrow{c} \cdot \overrightarrow{g} = 0$. This implies $-18 + 2\lambda - 8 = 0$ and therefore $\lambda = 13$.

(q) The length of $\overrightarrow{e}$ is $||\overrightarrow{e}|| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$.

(r) $\overrightarrow{d}$ and the x-axis enclose the angle $\theta = \arctan \frac{2}{1} = 63°$. Graphing $\overrightarrow{d}$ then shows that the angle between $\overrightarrow{d}$ and the positive x-axis is $180° - 63° = 117°$.

(s) The angle between $\overrightarrow{a}$ and $\overrightarrow{b}$ can be found as $\theta = \arccos \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{||\overrightarrow{a}||\ ||\overrightarrow{b}||} = \arccos \frac{-5}{\sqrt{14}\sqrt{17}} = 108.9°$.

(t) The equation of the plane perpendicular to $\overrightarrow{e}$ and passing through the point $(1, 2, 1)$ is $(x - 1) - (y - 2) + (z - 1) = 0$.

(u) The equation of the plane perpendicular to $\overrightarrow{b}$ and passing through the point $(2, 2, 0)$ is $-2(x - 2) - 3(y - 2) - 2(z - 0) = 0$.

(v) The area of the parallelogram with edges $\overrightarrow{a}$ and $\overrightarrow{d}$ is $||\overrightarrow{a} \times \overrightarrow{d}||$. Consider that
$\overrightarrow{a} \times \overrightarrow{d} = ((1)(0) - (-2)(2))\,\overrightarrow{i} + ((-2)(-1) - (3)(0))\,\overrightarrow{j} + ((3)(2) - (1)(-1))\,\overrightarrow{k} =$
$= (0 + 4)\,\overrightarrow{i} + (2 + 0))\,\overrightarrow{j} + (6 + 1))\,\overrightarrow{k} = 4\,\overrightarrow{i} + 2\,\overrightarrow{j} + 7\,\overrightarrow{k}$ to get
$||\overrightarrow{a} \times \overrightarrow{d}|| = \sqrt{4^2 + 2^2 + 7^2} = \sqrt{69}$.

(w) The volume of the parallelpiped with edges $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{e}$ is $|(\overrightarrow{a} \times \overrightarrow{b}) \cdot \overrightarrow{e}$. Using the determinant method to compute this quantity I get $|(\overrightarrow{a} \times \overrightarrow{b}) \cdot \overrightarrow{e}| = |-9-2-4-6-6+2| = 25$.

(x) A vector perpendicular to $\overrightarrow{a}$ and $\overrightarrow{e}$ is $\overrightarrow{a} \times \overrightarrow{e} = \overrightarrow{i} - 5\overrightarrow{j} - 4\overrightarrow{k}$. A unit vector in the same direction is $\overrightarrow{n} = \frac{\overrightarrow{a} \times \overrightarrow{e}}{||\overrightarrow{a} \times \overrightarrow{e}||} = \frac{1}{\sqrt{42}}(\overrightarrow{i} - 5\overrightarrow{j} - 4\overrightarrow{k})$.

3. Consider the points $P = (1, 2, 3)$, $Q = (3, 1, 4)$ and $R = (2, 5, 6)$. I will denote the vectors from the origin to one of these points with $\overrightarrow{p}$, $\overrightarrow{q}$ and $\overrightarrow{r}$ respectively.

(a) $\overrightarrow{PQ} = \overrightarrow{q} - \overrightarrow{p} = (3 - 1)\overrightarrow{i} + (1 - 2)\overrightarrow{j} + (4 - 3)\overrightarrow{k} = 2\overrightarrow{i} - \overrightarrow{j} + \overrightarrow{k}$.

(b) The distance between P and Q is just the length of the vector $\overrightarrow{PQ}$:
$||\overrightarrow{PQ}|| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$.

(c) Consider that $\overrightarrow{QR} = \overrightarrow{r} - \overrightarrow{q} = (2-3)\overrightarrow{i} + (5-1)\overrightarrow{j} + (6-4)\overrightarrow{k} = -\overrightarrow{i} + 4\overrightarrow{j} + 2\overrightarrow{k}$. Then the cosine of the angle PQR at vertex Q can be found as $\cos\angle PQR = \frac{\overrightarrow{PQ} \cdot \overrightarrow{QR}}{||\overrightarrow{PQ}||||\overrightarrow{QR}||} = \frac{-4}{\sqrt{6}\sqrt{21}}$.

(d) The angle between PQ and QR is $\angle PQR = \arccos \frac{-4}{\sqrt{6}\sqrt{21}} = 110.9°$.

(e) The area of the triangle PQR is half of the area of the parallelogram spanned by $\overrightarrow{PQ}$ and $\overrightarrow{QR}$. Therefore $A = \frac{1}{2}||\overrightarrow{PQ} \times \overrightarrow{QR}||$. Consider that $\overrightarrow{PQ} \times \overrightarrow{QR} = -6\,\overrightarrow{i} - 5\,\overrightarrow{j} + 7\,\overrightarrow{k}$ to get $A = \frac{1}{2}\sqrt{(-6)^2 + (-5)^2 + 7^2} = \frac{1}{2}\sqrt{110}$.

(f) $||\overrightarrow{PQ}|| = \sqrt{6}$
$||\overrightarrow{QR}|| = \sqrt{21}$
$||\overrightarrow{PR}|| = ||\overrightarrow{PQ} + \overrightarrow{QR}|| = \sqrt{19}$

(g) There are several ways to solve this problem:

- Taking $b = PQ$ as the base of the triangle PQR the distance from R to the line through P and Q is just the height h of the triangle. Since the area of the triangle has to satisfy both $A = \frac{1}{2}b \cdot h$ and $A = \frac{1}{2}\sqrt{110}$, it follows with $b = \sqrt{6}$ that $h = \sqrt{\frac{110}{6}}$.

- If one decomposes the vector $\overrightarrow{PR}$ into two vectors, one parallel to $\overrightarrow{PQ}$ and one perpendicular to $\overrightarrow{PQ}$, then the distance between R and the line through P and Q is just the length of the perpendicular vector. In this problem the $\overrightarrow{PR}$ can be decomposed into $\overrightarrow{PR}_{||} = \frac{4}{6}\overrightarrow{i} - \frac{2}{6}\overrightarrow{j} + \frac{2}{6}\overrightarrow{k}$ and $\overrightarrow{PR}_{\perp} = \frac{2}{6}\overrightarrow{i} + \frac{20}{6}\overrightarrow{j} + \frac{16}{6}\overrightarrow{k}$ and hence $||\overrightarrow{PR}_{\perp}|| = \sqrt{\frac{110}{6}}$.
- Instead of $\overrightarrow{PR}$ also $\overrightarrow{QR}$ could be decomposed.

(h) A vector perpendicular to a plane containing P, Q and R is in particular perpendicular to the vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ and hence one such vector is $\overrightarrow{PQ} \times \overrightarrow{QR} = -6\overrightarrow{i} - 5\overrightarrow{j} + 7\overrightarrow{k}$. A unit vector in the same direction is $\overrightarrow{n} = \frac{\overrightarrow{PQ}\times\overrightarrow{QR}}{||\overrightarrow{PQ}\times\overrightarrow{QR}||} = \frac{1}{10}\left(\overrightarrow{PQ} \times \overrightarrow{QR}\right) = -0.6\overrightarrow{i} - 0.5\overrightarrow{j} + 0.7\overrightarrow{k}$.

(i) An equation of the plane passing through the three points P, Q and R is $-6(x-1) - 5(y-2) + 7(z-3) = 0$.

4. The third vector $\overrightarrow{F_3}$ has to be chosen such that $\overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3} = \overrightarrow{0}$, i.e. $\overrightarrow{F_3} = -(\overrightarrow{F_1} + \overrightarrow{F_2})$. Since $\overrightarrow{F_1} + \overrightarrow{F_2} = 4\hat{\imath} + 3\hat{\jmath} - 2\hat{k} + 2\hat{\imath} + 3\hat{k} = 6\hat{\imath} + 3\hat{\jmath} + \hat{k}$ this yields $\overrightarrow{F_3} = -6\hat{\imath} - 3\hat{\jmath} - \hat{k}$.

5. In order to find the wind speed in direction of the dash, I need to find the component of $\overrightarrow{w}$ that is parallel to $\overrightarrow{v}$.
Consider that $||\overrightarrow{v}|| = \sqrt{3^2+4^2} = 5$. Therefore a unit vector in the same direction as $\overrightarrow{v}$ is $\overrightarrow{u} = \frac{3}{5}\overrightarrow{i} + \frac{4}{5}\overrightarrow{j}$. Then $\overrightarrow{w}_{||} = (\overrightarrow{w}\cdot\overrightarrow{u})\overrightarrow{u} = (2\cdot\frac{3}{5} + 6\cdot\frac{4}{5})\overrightarrow{u} = \frac{30}{5}\overrightarrow{u} = \frac{18}{5}\overrightarrow{i} + \frac{24}{5}\overrightarrow{j}$. This yields $||\overrightarrow{w}_{||}|| = \sqrt{\left(\frac{18}{5}\right)^2 + \left(\frac{24}{5}\right)^2} = \sqrt{\frac{900}{25}} = \sqrt{36} = 6$. Therefore the wind in direction of the dash exceeds the permitted wind speed.

6. In order to show, that $\overrightarrow{p}$ and $\overrightarrow{q}$ are orthogonal, I need to show that $\overrightarrow{p}\cdot\overrightarrow{q} = 0$. Consider that $\overrightarrow{p}\cdot\overrightarrow{q} = \overrightarrow{p}\cdot(\overrightarrow{x} - \overrightarrow{y}) = \overrightarrow{p}\cdot\overrightarrow{x} - \overrightarrow{p}\cdot\overrightarrow{y} = 5 - 5 = 0$. This proves the claim.

7. A vector $\overrightarrow{v}$ parallel to the line of intersection of the two planes $x - 2(y-1) + 3(z+2) = 0$ and $2(x-1) + (y+1) - 2(z-1) = 0$ has to be perpendicular to the normal vectors of both planes. Let $\overrightarrow{n_1}$ be a vector normal to the first plane and $\overrightarrow{n_2}$ a vector normal to the second plane. Then $\overrightarrow{n_1} = \overrightarrow{i} - 2\overrightarrow{j} + 3\overrightarrow{k}$ and $\overrightarrow{n_2} = 2\overrightarrow{i} + \overrightarrow{j} - 2\overrightarrow{k}$. A vector perpendicular to both $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ is the vector $\overrightarrow{n_1}\times\overrightarrow{n_2} = 4\overrightarrow{i} + 6\overrightarrow{j} + \overrightarrow{k} + 4\overrightarrow{k} - 3\overrightarrow{i} + 2\overrightarrow{i} = \overrightarrow{i} + 8\overrightarrow{j} + 5\overrightarrow{k}$. Hence a vector parallel to the intersection of the two planes is $\overrightarrow{v} = \overrightarrow{i} + 8\overrightarrow{j} + 5\overrightarrow{k}$.

8. (a) $\int e^{-x}dx$:
   - Substitution: $w = -x$
   - $\frac{dw}{dx} = -1 \Rightarrow dx = -dw$
   - $\int e^{-x}dx = \int e^{w}(-dw) = -\int e^{w}dw = -e^{w} + C = -e^{-x} + C$

   (b) $\int xe^{-x^2}dx$:

- Substitution: $w = -x^2$
- $\frac{dw}{dx} = -2x \Rightarrow dx = -\frac{1}{2x}dw$
- $\int xe^{-x^2}dx = \int xe^w(-\frac{1}{2x})dw = -\frac{1}{2}\int e^w dw = -\frac{1}{2}e^w + C = -\frac{1}{2}e^{-x^2} + C$

(c) $\int 4xe^{-4x^2}dx$:

- Substitution: $w = -4x^2$
- $\frac{dw}{dx} = -8x \Rightarrow dx = -\frac{1}{8x}dw$
- $\int 4xe^{-4x^2}dx = \int 4xe^w(-\frac{1}{8x})dw = -\frac{1}{2}\int e^w dw = -\frac{1}{2}e^w + C = -\frac{1}{2}e^{-4x^2} + C$

(d) $\int \frac{5e^{2x}}{1+e^{2x}}dx$:

- Substitution: $w = 1 + e^{2x}$
- $\frac{dw}{dx} = 2e^{2x} \Rightarrow dx = -\frac{1}{2e^{2x}}dw$
- $\int \frac{5e^{2x}}{1+e^{2x}}dx = \int \frac{5e^{2x}}{w}\frac{1}{2e^{2x}}dw = \frac{5}{2}\int \frac{1}{w}dw = \frac{5}{2}\ln|w| + C = \frac{5}{2}\ln|1+e^{2x}| + C =$
  $= \frac{5}{2}\ln(1+e^{2x}) + C$

(e) $\int \frac{e^x}{(e^x+a)^2}dx$:

- Substitution: $w = e^x + a$
- $\frac{dw}{dx} = e^x \Rightarrow dx = \frac{1}{e^x}dw$
- $\int \frac{e^x}{(e^x+a)^2}dx = \int \frac{e^x}{w^2}\frac{1}{e^x}dw = \int \frac{1}{w^2}dw = -\frac{1}{w} + C = -\frac{1}{e^x+a} + C$

(f) $\int \sin xe^{2\cos x}dx$:

- Substitution: $w = 2\cos x$
- $\frac{dw}{dx} = -2\sin x \Rightarrow dx = -\frac{1}{2\sin x}dw$
- $\int \sin xe^{2\cos x}dx = \int \sin xe^w(-\frac{1}{2\sin x})dw = -\frac{1}{2}\int e^w dw = -\frac{1}{2}e^w + C = -\frac{1}{2}e^{2\cos x} + C$

(g) $\int \sin\theta(\cos\theta + 2)^3 d\theta$:

- Substitution: $w = \cos\theta + 2$
- $\frac{dw}{d\theta} = -\sin\theta \Rightarrow d\theta = -\frac{1}{\sin\theta}dw$
- $\int \sin\theta(\cos\theta+2)^3 d\theta = \int \sin\theta w^3(-\frac{1}{\sin\theta})dw = -\int w^3 dw = -\frac{w^4}{4} + C =$
  $= -\frac{1}{4}(\cos\theta + 2)^4 + C$

(h) $\int \frac{e^t+1}{e^t+t}dt$:

- Substitution: $w = e^t + t$
- $\frac{dw}{dt} = e^t + 1 \Rightarrow dt = \frac{1}{e^t+1}dw$
- $\int \frac{e^t+1}{e^t+t}dt = \int \frac{e^t+1}{w}\frac{1}{e^t+1}dw = \int \frac{1}{w}dw = \ln|w| + C = \ln|e^t+t| + C$

(i) $\int \frac{1+2e^{2x}}{\sqrt{x+e^{2x}}}dx$:

- Substitution: $w = x + e^{2x}$
- $\frac{dw}{dx} = 1 + 2e^{2x} \Rightarrow dx = \frac{1}{1+2e^{2x}}dw$
- $\int \frac{1+2e^{2x}}{\sqrt{x+e^{2x}}}dx = \int \frac{1+2e^{2x}}{\sqrt{w}}\frac{1}{1+2e^{2x}}dw = \int \frac{1}{\sqrt{w}}dw = 2w^{1/2} + C = 2\sqrt{x+e^{2x}} + C$

(j) $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}dx$:

- Substitution: $w = e^x + e^{-x}$
- $\frac{dw}{dx} = e^x - e^{-x} \Rightarrow dx = \frac{1}{e^x-e^{-x}}dw$
- $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}dx = \int \frac{e^x-e^{-x}}{w}\frac{1}{e^x-e^{-x}}dw = \int \frac{1}{w}dw = \ln|w| + C = \ln|e^x+e^{-x}| + C =$
  $= \ln(e^x + e^{-x}) + C$

(k) $\int e^{-A\cos\theta} A\sin\theta d\theta$:

- Substitution: $w = -A\cos\theta$
- $\frac{dw}{d\theta} = A\sin\theta \Rightarrow d\theta = \frac{1}{A\sin\theta}dw$
- $\int e^{-A\cos\theta} A\sin\theta d\theta = \int e^w A\sin\theta \frac{1}{A\sin\theta}dw = \int e^w dw = e^w + C = e^{-A\cos\theta} + C$

(l) $\int \frac{5x-2}{\sqrt{x+1}}dx$:

- Substitution: $w = x+1$
- $\frac{dw}{dx} = 1 \Rightarrow dx = dw$
- $x = w-1$
- $\int \frac{5x-2}{\sqrt{x+1}}dx = \int \frac{5(w-1)-2}{\sqrt{w}}dw = \int \left(\frac{5w}{\sqrt{w}} - \frac{7}{\sqrt{w}}\right)dw = 5\int \sqrt{w}dw - 7\int \frac{1}{\sqrt{w}}dw =$
  $= 5\frac{w^{3/2}}{3/2} - 7\frac{w^{1/2}}{1/2} + C = \frac{10}{3}w^{3/2} - 14\sqrt{w} + C = \frac{10}{3}(x+1)^{3/2} - 14\sqrt{x+1} + C$

9. (a) $\int xe^{-x}dx$:

- $u(x) = x,\ v'(x) = e^{-x}$
- $\Rightarrow u'(x) = 1,\ v(x) = -e^{-x}$
- $\int xe^{-x}dx = x(-e^{-x}) - \int(-e^{-x})dx = -xe^{-x} + \int e^{-x}dx = xe^{-x} - e^{-x} + C$

(b) $\int t\cos t dt$:

- $u(t) = t,\ v'(t) = \cos t$
- $\Rightarrow u'(t) = 1,\ v(t) = \sin t$
- $\int t\cos t dt = t\sin t - \int \sin t dt = t\sin t + \cos t + C$

(c) $\int \ln x dx$:

- $u(x) = \ln x,\ v'(x) = 1$
- $\Rightarrow u'(x) = \frac{1}{x},\ v(x) = x$
- $\int \ln x dx = (\ln x)x - \int \frac{1}{x}x dx = x\ln x - \int 1 dx = x\ln x - x + C$

(d) $\int x^3 \ln x dx$:

- $u(x) = \ln x,\ v'(x) = x^3$
- $\Rightarrow u'(x) = \frac{1}{x},\ v(x) = \frac{1}{4}x^4$
- $\int x^3\ln x dx = (\ln x)\frac{1}{4}x^4 - \int \frac{1}{x}x^3 dx = \frac{1}{4}x^4\ln x - \int x^2 dx = \frac{1}{4}x^4\ln x - \frac{1}{3}x^3 + C$

(e) $\int 2x^2\sin x dx$:

- $u(x) = 2x^2,\ v'(x) = \sin x$
- $\Rightarrow u'(x) = 4x,\ v(x) = -\cos x$
- $\int 2x^2\sin x dx = 2x^2(-\cos x) - \int 4x(-\cos x)dx$
- Another partial integration is necessary to find $\int 4x(-\cos x)dx$.
- $U(x) = -4x,\ V'(x) = \cos x$
- $\Rightarrow U'(x) = -4,\ V(x) = \sin x$
- $\int -4x\cos x dx = -4x\sin x - \int -4\sin x dx = -4x\sin x + 4\int \sin x dx = -4x\sin x - 4\cos x + C$
- $\Rightarrow \int 2x^2\sin x dx = 2x^2(-\cos x) + 4x\sin x + 4\cos x + C$

(f) $\int 2\cos^2 t dt$:

- $u(t) = 2\cos t,\ v'(t) = \cos t$
- $\Rightarrow u'(t) = -2\sin t,\ v(t) = \sin t$

- $\int 2\cos^2 t dt = 2\cos t \sin t - \int(-2\sin t)\sin t dt = 2\cos t \sin t + 2\int \sin^2 t dt =$
  $= 2\cos t \sin t + 2\int(1-\cos^2 t)dt = 2\cos t \sin t + 2\int 1 dt - 2\int \cos^2 t dt =$
  $= 2\cos t \sin t + 2t + 2C - \int 2\cos^2 t dt$
- The original integral reappeared. Solving for $\int 2\cos^2 t dt$ now yields: $\int 2\cos^2 t dt = \cos t \sin t + t + C$

(g) $\int 2(\ln x)^2 dx$:

- $u(x) = 2(\ln x)^2,\ v'(x) = 1$
- $\Rightarrow u'(x) = 4\ln x \frac{1}{x},\ v(x) = x$
- $\int 2(\ln x)^2 dx = 2(\ln x)^2 x - \int 4\ln x \frac{1}{x} x dx = 2x(\ln x)^2 - 4\int \ln x dx$
- Using the result from problem (c) this yields: $\int 2(\ln x)^2 dx = 2x(\ln x)^2 - 4(x\ln x - x) + C$

(h) $\int \frac{t+8}{\sqrt{4-t}} dt$:

- $u(t) = t+8,\ v'(t) = (4-t)^{-1/2}$
- $\Rightarrow u'(t) = 1,\ v(t) = -\frac{(4-t)^{1/2}}{1/2} = -2(4-t)^{1/2}$
- $\int \frac{t+8}{\sqrt{4-t}} dt = -2(t+8)(4-t)^{1/2} - \int -2(4-t)^{1/2} dt = -2(t+8)(4-t)^{1/2} + 2\int(4-t)^{1/2} dt =$

  $= -2(t+8)(4-t)^{1/2} + 2\frac{(4-t)^{3/2}}{3/2} + C = -2(t+8)(4-t)^{1/2} + \frac{4}{3}(4-t)^{3/2} + C$

(i) $\int x^5 \cos(x^3) dx$:

- This problem needs to be solved in two steps: first it needs to be simplified using an appropriate substitution, then integration by parts can be applied.
- Substitution: $w = x^3$
- $\frac{dw}{dx} = 3x^2 \Rightarrow dx = \frac{1}{3x^2} dw$
- $\int x^5 \cos(x^3) dx = \int x^5 (\cos w) \frac{1}{3x^2} dw = \frac{1}{3}\int x^3 \cos w dw = \frac{1}{3}\int w \cos w dw$
- Integration by parts:
  - $u(w) = w,\ v'(t) = \cos w$
  - $\Rightarrow u'(w) = 1,\ v(w) = \sin w$
  - $\int x^5 \cos(x^3) dx = \frac{1}{3}\int w \cos w dw = \frac{1}{3} w \sin w - \frac{1}{3}\int \sin w dw =$
    $= \frac{1}{3} w \sin w - \frac{1}{3}(-\cos w) + C = \frac{1}{3} w \sin w + \frac{1}{3}\cos w + C$
- Resubstition: $\int x^5 \cos(x^3) dx = \frac{1}{3} w \sin w + \frac{1}{3}\cos w + C = \frac{1}{3} x^3 \sin(x^3) + \frac{1}{3}\cos(x^3) + C$

(j) $\int \ln(1+t) dt$:

- This problem needs to be solved in two steps: first it needs to be simplified using integration by parts, then it can be solved using an appropiate substitution.
- $u(t) = \ln(1+t),\ v'(t) = 1$
- $\Rightarrow u'(t) = \frac{1}{1+t},\ v(t) = t$
- $\int \ln(1+t) dt = \ln(1+t)t - \int \frac{1}{1+t} t dt$
- $\int \frac{1}{1+t} t dt$:
  - Substitution: $w = 1+t$
  - $\frac{dw}{dx} = 1 \Rightarrow dx = dw$
  - $t = w - 1$
  - $\int \frac{1}{1+t} t dt = \int \frac{t}{w} dw = \int \frac{w-1}{w} dw = \int(1 - \frac{1}{w}) dw = w - \ln|w| + C =$
    $= 1 + t - \ln|1+t| + C$

- $\int \ln(1+t)dt = \ln(1+t)t - \int \frac{1}{1+t}tdt = \ln(1+t)t - 1 - t + \ln(1+t) + C$

(k) $\int \arcsin t dt$:

- This problem needs to be solved in two steps: first it needs to be simplified using integration by parts, then it can be solved using an appropiate substitution.
- $u(t) = \arcsin t,\ v'(t) = 1$
- $\Rightarrow u'(t) = \frac{1}{\sqrt{1-t^2}},\ v(t) = t$
- $\int \arcsin t dt = (\arcsin t)t - \int \frac{1}{1-t^2}tdt = (\arcsin t)t - \int \frac{t}{1-t^2}dt$
- $\int \frac{t}{\sqrt{1-t^2}}dt$:
  - Substitution: $w = 1 - t^2$
  - $\frac{dw}{dt} = -2t \Rightarrow dt = -\frac{1}{2t}dw$
  - $\int \frac{t}{\sqrt{1-t^2}}dt = \int \frac{t}{\sqrt{w}}(-\frac{1}{2t})dw = -\frac{1}{2}\int \frac{1}{\sqrt{w}}dw = -\frac{1}{2}\frac{w^{1/2}}{1/2} + C =$
    $= \sqrt{w} + C = \sqrt{1-t^2} + C$
- $\int \arcsin t dt = (\arcsin t)t - \int \frac{t}{1-t^2}dt = (\arcsin t)t - \sqrt{1-t^2} + C$

10. Let f be twice differentiable with $f(1) = 2$, $f(2) = 5$, $f'(1) = 1$ and $f'(2) = 3$. Setting $u(x) = x$, $v'(x) = f''(x)$ yields $u'(x) = 1$ and $v(x) = f'(x)$ and hence:
$\int_1^2 xf''(x)dx = xf'(x)|_1^2 - \int_1^2 f'(x)dx = xf'(x)|_1^2 - f(x)|_1^2 = (2\cdot f'(2) - 1\cdot f'(1)) - (f(2) - f(1)) =$
$= 6 - 1 - 5 + 2 = 2.$

11. (a) $\int \frac{1}{(x-2)(x-3)}dx$:

- $\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} \Rightarrow 1 = A(x-3) + B(x-2) \Rightarrow 1 = x(A+B) + (-3A-2B)$ $\Rightarrow A + B = 0$ and $-3A - 2B = 1$.
- $A + B = 0 \Rightarrow A = -B$
- $-3A - 2B = 1 \Rightarrow 3B - 2B = 1 \Rightarrow B = 1 \Rightarrow A = -1.$
- $\int \frac{1}{(x-2)(x-3)}dx = -\int \frac{1}{x-2}dx + \int \frac{1}{x-3}dx = -\ln|x-2| + \ln|x-3| + C$

(b) $\int \frac{x+1}{x^2+2x}dx$:

- $\frac{x+1}{x^2+2x} = \frac{x+1}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2} \Rightarrow x + 1 = A(x+2) + Bx \Rightarrow x + 1 = x(A+B) + 2A \Rightarrow$ $A + B = 1$ and $2A = 1$.
- $2A = 1 \Rightarrow A = \frac{1}{2}$
- $A + B = 1 \Rightarrow B = \frac{1}{2}$.
- $\int \frac{x+1}{x^2+2x}dx = \frac{1}{2}\int \frac{1}{x}dx + \frac{1}{2}\int \frac{1}{x+2}dx = \frac{1}{2}\ln|x| + \frac{1}{2}\ln|x+2| + C$

(c) $\int \frac{8x-4x^2}{(x-1)^2(x+3)}dx$:

- $\frac{8x-4x^2}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3} \Rightarrow 8x - 4x^2 = A(x-1)(x+2) + B(x+3) + C(x-1)^2$
  $\Rightarrow 8x - 4x^2 = A(x^2 + x - 2) + Bx + 3B + C(x^2 - 2x + 1)$
  $\Rightarrow 8x - 4x^2 = x^2(A+C) + x(A+B-2C) - 2A + 3B + C \Rightarrow -4 = A + C, 8 = A + B - 2C$ and $0 = -2A + 3B + C$.
- $-4 = A + C \Rightarrow A = -4 - C$
- $8 = A + B - 2C \Rightarrow 8 = -4 - C + B - 2C = -4 + B - 3C \Rightarrow B = 12 + 3C.$
- $0 = -2A + 3B + C \Rightarrow 0 = -2(4 - C) + 3(12 + 3C) + C = -8 + 2C + 36 + 9C + C =$ $28 + 12C \Rightarrow C = -\frac{28}{12} = -\frac{7}{3}.$

- $\Rightarrow A = -4 - C = -\frac{5}{3}$, $B = 12 + 3C = 12 - 7 = 5$.
- $\int \frac{8x-4x^2}{(x-1)^2(x+3)}dx = -\frac{5}{3}\int \frac{1}{x-1}dx + 5\int \frac{1}{(x-1)^2}dx - \frac{7}{3}\int \frac{1}{x+3}dx =$
  $= -\frac{5}{3}\ln|x-1| + 5\frac{(x-1)^{-1}}{-1} - \frac{7}{3}\ln|x+3| + C = -\frac{5}{3}\ln|x-1| - 5\frac{1}{x-1} - \frac{7}{3}\ln|x+3| + C$

(d) $\int \frac{x^2-2x-\frac{1}{3}}{(x^2+1)(x-3)}dx$:

- $\frac{x^2-2x+\frac{1}{3}}{(x^2+1)(x-3)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-3} \Rightarrow x^2 - 2x - \frac{1}{3} = (Ax+B)(x-3) + C(x^2+1)$
  $\Rightarrow x^2 - 2x - \frac{1}{3} = Ax^2 - 3Ax + Bx - 3B + Cx^2 + C = x^2(A+C) + x(-3A+B) - 3B + C$
  $\Rightarrow 1 = A + C$, $-2 = -3A + B$ and $\frac{1}{3} = -3B + C$.
- $1 = A + C \Rightarrow A = 1 - C$
- $-2 = -3A + B = -3(1-C) + B = -3 + 3C + B \Rightarrow B = 1 - 3C$.
- $\frac{1}{3} = -3B + C \Rightarrow \frac{1}{3} = -3(1-3C) + C = -3 + 10C \Rightarrow C = \frac{1}{3}$.
- $\Rightarrow A = 1 - C = \frac{2}{3}$, $B = 1 - 3C = 0$.
- $\int \frac{x^2-3x+\frac{1}{3}}{(x^2+1)(x-3)}dx = \frac{2}{3}\int \frac{x}{x^2+1}dx + \frac{1}{3}\int \frac{1}{x-3}dx = \frac{1}{3}\left(\int \frac{2x}{x^2+1}dx + \int \frac{1}{x-3}dx\right)$
- To determine $\int \frac{2x}{x^2+1}dx$ integration by substitution is necessary:
  - Substitution: $w = x^2 + 1$
  - $\frac{dw}{dx} = 2x \Rightarrow dx = \frac{1}{2x}dw$
  - $\int \frac{2x}{x^2+1}dx = \int \frac{2x}{w}\frac{1}{2x}dw = \int \frac{1}{w}dw = \ln|w| + C = \ln(x^2+1) + C$
- Plugging this into the above expression yields:
  $\int \frac{x^2-3x+\frac{1}{3}}{(x^2+1)(x-3)}dx = \frac{1}{3}\left(\ln(x^2+1) + \ln|x-3|\right) + C$

(e) $\int \frac{x^4-2x^3+x-1}{x^2-2x}dx$:

- Long division yields: $\frac{x^4-2x^3+x-1}{x^2-2x} = \frac{x^2(x^2-2x)+x-1}{x^2-2x} = x^2 + \frac{x-1}{x^2-2x} = x^2 + \frac{x-1}{x(x-2)}$
- $\Rightarrow \int \frac{x^4-2x^3+x-1}{x^2-2x}dx = \int x^2 dx + \int \frac{x-1}{x(x-2)}dx = \frac{x^3}{3} + \int \frac{x-1}{x(x-2)}dx$.
- To integrate $\frac{x-1}{x(x-2)}$ the method of partial fractions needs to be applied:
  - $\frac{x-1}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2} \Rightarrow x - 1 = A(x-2) + Bx = x(A+B) - 2A \Rightarrow A + B = 1$, $-1 = -2A$.
  - $-1 = -2A \Rightarrow A = \frac{1}{2}$
  - $A + B = 1 \Rightarrow B = \frac{1}{2}$
  - $\int \frac{x-1}{x(x-2)} = \frac{1}{2}\int \frac{1}{x}dx + \frac{1}{2}\int \frac{1}{x-2}dx = \frac{1}{2}\ln|x| + \frac{1}{2}\ln|x-2| + C$
- Plugging this into the above expression yields:
  $\int \frac{x^4-2x^3+x-1}{x^2-2x}dx = \frac{x^3}{3} + \int \frac{x-1}{x(x-2)}dx = \frac{x^3}{3} + \frac{1}{2}\ln|x| + \frac{1}{2}\ln|x-2| + C$.

(f) $\int \frac{1}{x^2-4}dx$:

- $\frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} \Rightarrow 1 = A(x+2) + B(x-2) = Ax + 2A + Bx - 2B =$
  $= x(A+B) + 2A - 2B \Rightarrow A + B = 0$, $2A - 2B = 1$
- $A + B = 0 \Rightarrow A = -B$
- $2A - 2B = 1 \Rightarrow -4B = 1 \Rightarrow B = -\frac{1}{4} \Rightarrow A = \frac{1}{4}$
- $\Rightarrow \frac{1}{x^2-4} = \frac{1}{4}\frac{1}{x-2} - \frac{1}{4}\frac{1}{x+2}$
- $\int \frac{1}{x^2-4}dx = \frac{1}{4}\int \frac{1}{x-2}dx - \frac{1}{4}\int 1x + 2dx = \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C$

(g) $\int \frac{3}{4x-x^2}dx$:

- $\frac{3}{4x-x^2} = \frac{3}{x(4-x)} = -\frac{3}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4} \Rightarrow -3 = A(x-4) + Bx = x(A+B) - 4A$
  $\Rightarrow A + B = 0,\ -4A = -3$
- $-4A = -3 \Rightarrow A = \frac{3}{4}$
- $A + B = 0 \Rightarrow B = -\frac{3}{4}$
- $\Rightarrow \frac{3}{4x-x^2} = \frac{3}{4}\frac{1}{x} - \frac{3}{4}\frac{1}{x-4}$
- $\int \frac{3}{4x-x^2}dx = \frac{3}{4}\int \frac{1}{x}dx - \frac{3}{4}\int 1x - 4dx = \frac{3}{4}\ln|x| - \frac{3}{4}\ln|x-4| + C$

(h) $\int \frac{x^5-2x^4+x^3-3x-1}{x^2-2x+1}dx$:

- Long division yields:
  $\frac{x^5-2x^4+x^3-3x-1}{x^2-2x+1} = \frac{x^3(x^2-2x+1)-3x-1}{x^2-2x+1} = x^3 - \frac{3x+1}{x^2-2x+1} = x^3 - \frac{3x+1}{(x-1)^2}$
- $\Rightarrow \int \frac{x^5-2x^4+x^3-3x-1}{x^2-2x+1}dx = \int x^3dx - \int \frac{3x+1}{(x-1)^2}dx = \frac{x^4}{4} - \int \frac{3x+1}{(x-1)^2}dx.$
- To integrate $\frac{3x+1}{(x-1)^2}$ the method of partial fractions needs to be applied:
  - $\frac{3x+1}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 3x + 1 = A(x-1) + B = Ax - A + B$
  - $\Rightarrow A = 3,\ -A + B = 1 \Rightarrow B = 4$
  - $\int \frac{3x-1}{(x-1)^2} = \int \frac{3}{x-1}dx + \int \frac{4}{(x-1)^2}dx = 3\ln|x-1| - 4\frac{1}{x-1} + C$
- Plugging this into the above expression yields:
  $\int \frac{x^5-2x^4+x^3-3x-1}{x^2-2x+1}dx = \frac{x^4}{4} - \int \frac{3x+1}{(x-1)^2}dx = \frac{x^4}{4} - 3\ln|x-1| + 4\frac{1}{x-1} + C$